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HOMOGENIZING STIFFNESSES OF PLATES WITH PERIODIC STRUCTURE

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Abstract—The paper develops the method of homogenization of elastic properties of Reissner–Hencky plates with periodic structure. The method is based on the new scaling which for each ε preserves the shape of the three-dimensional periodicity cell of the original plate structure. The homogenization formulae obtained in this manner turn out to coincide with those found previously by the Hencky–Reissner approximation of the three-dimensional Caillerie–Kohn–Vogelius local problems. The proposed method has made it possible to derive a new closed-form formula for evaluating the effective torsional stiffness of plates with one family of stiffeners (the formulae for remaining stiffnesses have been known). Contrary to the hitherto-used formula of Duvaut, and to the one followed from a direct homogenization of the Reissner–Hencky equations, the derived new formula is sensitive to the thickness-to-period ratio and hence much more realistic. The formulae used previously turn out to be the upper and lower bounds for the derived expression.

1. INTRODUCTION

In the problem of homogenization of stiffnesses a plate with periodic structure can be treated as:

(i) a three-dimensional body (Caillerie, 1984; Kohn and Vogelius, 1984, 1985, 1986; Kalamkarov *et al.*, 1987; Reztsov, 1990; Lewiński, 1991a–d; Chacha and Sanchez-Palencia, in press)

or its deformations can be described by

(ii) two-dimensional plate models:
- (ii.1) the Kirchhoff model (Duvaut, 1976; Duvaut and Metellus, 1976; Kolpakov, 1982; Lewiński and Telega, 1988a; Lewiński, 1991c)
- (ii.2) the model of Reissner–Hencky (Bourgeat and Tapiéro, 1983; Tadlaoui and Tapiéro, 1988; Lewiński and Telega, 1988b; Telega and Lewiński, 1988; Lewiński, 1991c),

or by other models of higher order (Lewiński and Telega, 1989; in press; Lewiński, 1991c).

In the case of approach (i) the problem considered is $\mathscr{Z}$-periodic, $\mathscr{Z}$ being a three-dimensional cell of periodicity. The first stage of the homogenization method is to extrapolate an ε-indexed family (P_ε) of $\mathscr{Z}^\varepsilon$-periodic problems through a given problem $(P_{\mathscr{Z}})$ with $\mathscr{Z}$-periodic material or geometrical characteristics. For a certain ε_0, $(P_{\varepsilon_0}) = (P_{\mathscr{Z}})$ and $\mathscr{Z}^{\varepsilon_0} = \mathscr{Z}$. The formation of the (P_ε) family of $\mathscr{Z}^\varepsilon$-periodic coefficients is built upon the following assumption of similarity:

$$\mathscr{Z}^\varepsilon = \varepsilon\mathscr{Y}, \tag{1}$$

where $\mathscr{Y}$ is referred to as a rescaled cell and $\mathscr{Z} = \varepsilon_0\mathscr{Y}$. One assumes usually that $\varepsilon_0 = 1$. The global dimensions of the plate domain are kept as ε-independent. The asymptotic homogenization that starts from description (i) and is based on the scaling (1) results in the two-dimensional Kirchhoff-type homogenized model of the plate of stiffnesses determined by some auxiliary functions being solutions to the three-dimensional, so-called basic cell problems posed on $\mathscr{Y}$. This approach is well-substantiated. However, the essential drawback of it is that these basic cell problems are difficult to solve both analytically and numerically. Thus it seems justified to base the homogenization process on one of two-dimensional plate models.

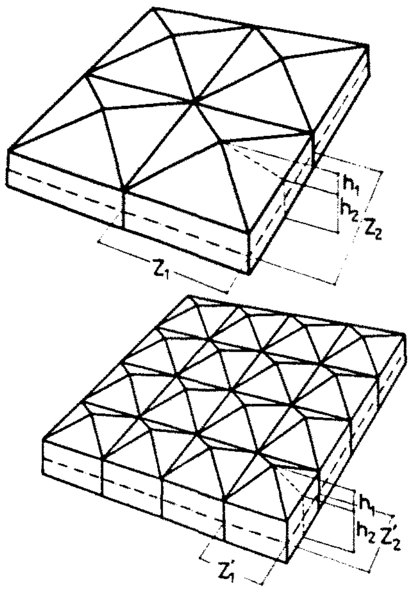

Fig. 1. For these plates the formulae for the effective stiffnesses based on the in-plane scaling are identical.

Within the framework of the Kirchhoff description (ii.1) the elastic properties of a symmetric plate in bending are characterized by one stiffness tensor $D^{\alpha\beta\lambda\mu}$. In the case of plates with periodic structure these stiffnesses are Z-periodic, $Z = (0, Z_1) \times (0, Z_2)$ being a reference plane for the cell $\mathscr{Y}$. The approach (ii.1) is based on the scaling

$$Z^{\varepsilon} = \varepsilon Y, \quad Z^{\varepsilon_0} = Z, \tag{2}$$

where Z^{ε} is a rectangular period of the variation of stiffnesses $D_{\varepsilon}^{\alpha\beta\lambda\mu}$ of the plate in the (P_{ε}) problem; Y is a rescaled rectangle $(0, Y_1) \times (0, Y_2)$ of periodicity. Scaling (2), which is a two-dimensional counterpart of scaling (1), will be referred to as an in-plane scaling. The first homogenization formulae based on the in-plane scaling were found in Duvaut and Metellus (1976) and in Duvaut (1976). The essential drawback of these formulae is that they do not distinguish between plates of different ratios Z_{α}/h, h being an average plate thickness (cf. Fig. 1). The reason for this is that the Kirchhoff model does not convey the information that the parameter h, concealed in the $D^{\alpha\beta\lambda\mu}$ tensor, represents the plate thickness. In other words there are no length scales in the Kirchhoff model and the parameter h is not a length scale of the model either. Thence arises a need for improving the formulae of Duvaut to make them sensitive to the Z_{α}/h ratios.

The method (ii.2) seems to be a natural improvement of the (ii.1) approach. If one assumes the Reissner–Hencky model of the plate with Z-periodic bending ($D^{\alpha\beta\lambda\mu}$) and shearing ($H^{\alpha\beta}$) stiffnesses as the point of departure, the (P_{ε}) family can be formed twofold by

(ii.2.a) assuming in-plane scaling (eqn (2)), or
(ii.2.b) putting

$$Z^{\varepsilon} = \varepsilon Y, \quad h^{\varepsilon} = \frac{\varepsilon}{\varepsilon_0} h. \tag{3}$$

Version (ii.2.a) has for the first time been used in Bourgeat and Tapiéro (1983). Within the framework of this approach the effective model assumes the form of the Reissner–Hencky plate model with constant, averaged stiffnesses. The local problem splits up into two independent, bending and shearing, local problems (cf. Lewiński, 1991c, Section 3). However, both of them turn out to be free from their own natural length scales and, consequently, the effective stiffnesses become insensitive to the Z_α/h ratios. Thus the formulae for effective stiffnesses are stricken by the same drawback as the formulae of Duvaut.

The aim of the present paper is to amend the approach of Bourgeat and Tapiéro (1983) by assuming scaling (3). This refined scaling can be viewed as a consequence of scaling (1) used in the three-dimensional approach. Note, however, that the plate thickness h does not enter the Reissner–Hencky equations explicitly. Thus the second part of scaling (3) should be expressed in terms of the stiffnesses $D^{\alpha\beta\lambda\mu}$ and $H^{\alpha\beta}$. Let us define

$$l_j = (D^{\alpha\beta\lambda\mu}/H^{\sigma\sigma})^{1/2}, \quad j = (\alpha, \beta, \lambda, \mu, \sigma). \tag{4}$$

The quantities l_j are natural length scales of the Reissner–Hencky plate model. Since l_j are proportional to h, scaling (3) can be rewritten as follows:

$$Z^\varepsilon = \varepsilon Y, \quad l_j^\varepsilon = \frac{\varepsilon}{\varepsilon_0} l_j. \tag{5}$$

Note that the mutual relations between the length scales Z_α^ε, l_j^ε involved in the problem are ε-independent. Note, moreover, that scaling (5) can be viewed as a generalization of scaling (1) since in the three-dimensional model of elasticity the length scales are absent.

Considerations of the present paper will be confined to the case of plates symmetric with respect to the middle plane and subjected to transverse loads; therefore only the bending/shearing problem will be dealt with. With the help of the asymptotic analysis (ii.2.b) we arrive at the effective model of Kirchhoff type. The local problems turn out to be of the Reissner–Hencky type and the bending and shearing effects do not decouple. Consequently, the solutions to these problems and the effective stiffnesses determined by them become dependent on the thickness-to-period ratios h/Z_α.

The formulae derived with the help of the refined scaling turn out to coincide with those previously found in Lewiński (1991c, Section 5) via Hencky–Reissner reduction of the transverse dimension of the three-dimensional Caillerie–Kohn–Vogelius local problem. Thus by virtue of introducing the refined scaling (5) it was possible to show that two operations—homogenizing elastic properties and Hencky–Reissner reduction of the transverse dimension—commute (cf. Diagram 1 in Section 4). On the other hand, the asymptotic method used in Lewiński (1991c, Section 3) was based on the in-plane scaling, which led us to Diagram 2 (*ibid.*) that lacked this property. The present paper explains that the noncommutativeness was due to inconsistency between the asymptotic methods used in the left- and right-hand sides of that diagram.

The discrepancies between various methods of homogenization can be examined by comparing the related formulae for effective stiffnesses of plates of thickness varying periodically in one direction. The formulae for the effective bending stiffnesses $D^{\alpha\alpha\beta\beta}$ are beyond argument, since all two-dimensional homogenization algorithms lead to identical equations, shown for the first time by Duvaut (1976). In Lewiński (1991c) there has been noted an essential discrepancy between the predictions for the effective torsional stiffness D^{1212} according to approaches (ii.1), (ii.2.a) and (ii.2.b). The formulae for D^{1212} found by the first two methods are not sensitive to the length of the period measured with respect to the plate transverse dimensions. Some numerical results of D^{1212} predicted by method (ii.2.b) have been reported in Lewiński (1991c, Section 7). In the present paper there will be derived a simple closed-form expression for D^{1212} depending explicitly upon all coefficients determining the shape of the plate with rapidly varying thickness. Moreover, there will be given a proof that the methods (ii.1) and (ii.2.a) provide us with the upper and lower bounds for the (ii.2.b) predictions.

Throughout the paper, a conventional summation convention is adopted. The Greek indices (except for ε) run over 1, 2. Moreover, the following symbols of differentiation will be used: $\partial/\partial x_\alpha = {}_{,\alpha}$, $\partial/\partial y_\alpha = {}_{|\alpha}$. The brackets $\{\cdot\}$ mean averaging over the rectangle Y.

2. REISSNER-HENCKY MODEL OF PLATES WITH PERIODIC STRUCTURE

Consider an elastic plate symmetric with respect to its middle plane Ω, parametrized by the Cartesian coordinates $\mathbf{x} = (x, x_3)$, $x = (x_\alpha)$. The elastic moduli $C_Z^{ijkl}(\mathbf{x})$ (where i, j, k, l run over 1, 2, 3) are even functions in x_3, and the planes x_3 = constant are planes of material symmetry, viz.

$$C_Z^{3\alpha\beta\gamma} = C_Z^{333\delta} = 0. \tag{6}$$

The moduli C_Z^{ijkl} are assumed to be Z-periodic in x, $Z = (0, Z_1) \times (0, Z_2)$ being a rectangle. The thickness of the plate varies Z-periodically in x and evenly in x_3, hence

$$-x_3^+(x) \leqslant x_3 \leqslant x_3^+(x)$$

and $x_3^+(\cdot)$ is Z-periodic. The plate is clamped along its lateral edge. The faces $x_3 = \pm x_3^+$ are subjected to the transverse loading $p_3^{Z\pm}(x, x)$; the functions $p_3^{Z\pm}(x, \cdot)$ are Z-periodic. Let us define

$$G_Z(x) = 1 + (x_{3,1}^+)^2 + (x_{3,2}^+)^2. \tag{7}$$

The further analysis will be confined to the bending problem, which, by virtue of the transverse symmetry of geometry and elastic properties, separates from the membrane phenomena. The bending and shearing stiffnesses are defined as follows:

$$D_Z^{\alpha\beta\lambda\mu}(x) = \int_{-x_3^+(x)}^{x_3^+(x)} (x_3)^2 \tilde{C}_Z^{\alpha\beta\lambda\mu}(x, x_3)\,\mathrm{d}x_3,$$

$$H_Z^{\alpha\beta}(x) = \kappa \int_{-x_3^+(x)}^{x_3^+(x)} C_Z^{3\alpha3\beta}(x, x_3)\,\mathrm{d}x_3 \tag{8}$$

where κ is a shear correction factor (we shall assume that $\kappa = 5/6$) and the tensor $\tilde{\mathbf{C}}_Z$ given by

$$\tilde{C}_Z^{\alpha\beta\lambda\mu} = C_Z^{\alpha\beta\lambda\mu} - C_Z^{\alpha\beta33}C_Z^{33\lambda\mu}/C_Z^{3333} \tag{9}$$

is related to the plane-stress approximation.

Within the framework of the Reissner–Hencky plate model the unknown fields are (w, φ); w represents the averaged transverse deflection and $\varphi = (\varphi_\alpha)$ are the averaged rotations of the transverse cross-sections. The space of kinematically admissible fields reads $V(\Omega) = H_0^1(\Omega) \times [H_0^1(\Omega)]^2$. The equilibrium problem assumes the form

$$(P_{HR}) \left| \begin{array}{l} \text{find } (w, \varphi) \in V(\Omega) \text{ such that} \\ \displaystyle\int_\Omega [D_Z^{\alpha\beta\lambda\mu}\varphi_{\alpha,\beta}\psi_{\lambda,\mu} + H_Z^{\alpha\beta}(\varphi_\alpha + w_{,\alpha})(\psi_\beta + v_{,\beta})]\,\mathrm{d}x = \int_\Omega q^Z v\,\mathrm{d}x \\ \text{for every } (v, \psi) \in V(\Omega), \end{array}\right.$$

where

$$q^Z(x) = q_Z(x, x), \quad q_Z(x, y) = (p_3^{Z+} + p_3^{Z-})(x, y)[G_Z(y)]^{1/2}.$$

The problem (P_{HR}) is uniquely solvable (cf. Lagnese and Lions, 1988).

3. HOMOGENIZATION BASED UPON THE IN-PLANE SCALING

Since for small Z_α the problem (P_{HR}) is intractable even by numerical methods it is useful to replace this problem by an effective one with smeared-out coefficients. Such an effective problem can be found by the homogenization method built on the in-plane scaling (eqn (2)). The relevant family P^ε_{HR} is constructed by replacing

$$x_3^+(x) \to h\left(\frac{x}{\varepsilon}\right) = x_3^+(x\cdot\varepsilon_0/\varepsilon), \quad Z \to \varepsilon Y = \frac{\varepsilon}{\varepsilon_0}\cdot Z,$$

$$Y = (0, Y_1)\times(0, Y_2), \quad G_Z(x) \to G\left(\frac{x}{\varepsilon}\right) = G_Z(x\varepsilon_0/\varepsilon),$$

$$C_Z^{ijkl}(\mathrm{x}) \to \tilde{C}^{ijkl}\left(\frac{x}{\varepsilon}, x_3\right) = C_Z^{ijkl}\left(\frac{x\varepsilon_0}{\varepsilon}, x_3\right),$$

$$q^Z(x) \to q^\varepsilon(x) = q_Z(x, x\varepsilon_0/\varepsilon) = (p_3^+ + p_3^-)\left(x, \frac{x}{\varepsilon}\right)\left[G\left(\frac{x}{\varepsilon}\right)\right]^{1/2},$$

$$p_3^\pm(x, y) = p_3^{Z\pm}(x, \varepsilon_0 y). \tag{10}$$

Thus we substitute

$$D_Z^{\alpha\beta\lambda\mu}(x) \to \tilde{D}^{\alpha\beta\lambda\mu}\left(\frac{x}{\varepsilon}\right), \quad H_Z^{\alpha\beta}(x) \to H^{\alpha\beta}\left(\frac{x}{\varepsilon}\right), \tag{11}$$

where

$$\tilde{D}^{\alpha\beta\lambda\mu}(y) = \int_{-h(y)}^{h(y)} (x_3)^2 \tilde{\tilde{C}}^{\alpha\beta\lambda\mu}(y, x_3)\,\mathrm{d}x_3,$$

$$H^{\alpha\beta}(y) = \kappa \int_{-h(y)}^{h(y)} \tilde{C}^{\alpha3\beta3}(y, x_3)\,\mathrm{d}x_3 \tag{12}$$

and the tensor $\tilde{\tilde{\mathbf{C}}}$ is defined with the help of $\tilde{\mathbf{C}}$, according to rule (9). The functions $h(\cdot)$, $G(\cdot), p_3^\pm(x, \cdot), \tilde{C}^{ijkl}(\cdot, x_3), \tilde{D}^{\alpha\beta\lambda\mu}(\cdot)$ and $H^{\alpha\beta}(\cdot)$ are Y-periodic. For further details, see Lewiński (1991c, Section 3). The homogenization formulae (21)–(23) from Lewiński (1991c) based on scalings (10) and (11) turn out to be not affected by the ratios $Y_\alpha/h_{\max}$, which prompts one to improve the scaling.

4. HOMOGENIZATION BASED ON THE SCALING PRESERVING BENDING-TO-SHEARING STIFFNESS RATIOS

4.1. *Refined scaling*

The model (P_{HR}) involves the natural length scales defined by $(D_Z^{\alpha\beta\lambda\mu}/H_Z^{\sigma\sigma})^{1/2}$, and the length scales Z_α induced by periodicity of the problem. The ε-indexed family of $(\hat{P}^\varepsilon_{HR})$ problems will be formed so that $(\hat{P}^{\varepsilon_0}_{HR}) = (P_{HR})$ and for each ε the mutual relations between length scales will be the same. This can be achieved by assuming scaling (3). To this end we make the following substitutions:

$$Z \to \varepsilon Y = \frac{\varepsilon}{\varepsilon_0} Z, \quad G_Z(x) \to G\left(\frac{x}{\varepsilon}\right) = G_Z(x\varepsilon_0/\varepsilon),$$

$$x_3^+(x) \to \varepsilon c\left(\frac{x}{\varepsilon}\right) = \frac{\varepsilon}{\varepsilon_0} x_3^+(x\varepsilon_0/\varepsilon),$$

$$C_Z^{ijkl}(\mathbf{x}) \to C^{ijkl}\left(\frac{x_3}{\varepsilon}, \frac{x}{\varepsilon}\right) = C_Z^{ijkl}\left(\frac{x\varepsilon_0}{\varepsilon}, \frac{x_3\varepsilon_0}{\varepsilon}\right),$$

$$q^Z(x) \to \varepsilon^3 \hat{q}\left(x, \frac{x}{\varepsilon}\right), \quad \hat{q}(x,y) = (q^+ + q^-)(x,y)[G(y)]^{1/2},$$

$$q^{\pm}(x,y) = \frac{1}{\varepsilon_0^3} p_3^{\pm}(x,y). \tag{13}$$

The functions $C^{ijkl}(x_3/\varepsilon, \cdot)$, $c(\cdot)$, $q^{\pm}(x, \cdot)$ are Y-periodic.

New stiffnesses are defined according to their definition (8) and replacement (13). Hence

$$D_Z^{\alpha\beta\lambda\mu}(x) \to D^{\alpha\beta\lambda\mu}\left(\frac{x}{\varepsilon}\right) = \varepsilon^3 \hat{D}^{\alpha\beta\lambda\mu}\left(\frac{x}{\varepsilon}\right), \quad H_Z^{\alpha\beta}(x) \to \varepsilon \hat{H}^{\alpha\beta}\left(\frac{x}{\varepsilon}\right), \tag{14}$$

where

$$\hat{D}^{\alpha\beta\lambda\mu}(y) = \int_{-c(y)}^{c(y)} (y_3)^2 \tilde{C}^{\alpha\beta\lambda\mu}(y_3, y)\, \mathrm{d}y_3$$

$$\hat{H}^{\alpha\beta}(y) = \kappa \int_{-c(y)}^{c(y)} C^{\alpha 3\beta 3}(y_3, y)\, \mathrm{d}y_3. \tag{15}$$

The stiffnesses (12) and (15) are interrelated by

$$\tilde{D}^{\alpha\beta\lambda\mu}(y) = \varepsilon_0^3 \hat{D}^{\alpha\beta\lambda\mu}(y), \quad H^{\alpha\beta}(y) = \varepsilon_0 \hat{H}^{\alpha\beta}(y), \quad D^{\alpha\beta\lambda\mu}(y) = (\varepsilon/\varepsilon_0)^3 \tilde{D}^{\alpha\beta\lambda\mu}(y). \tag{16}$$

For $\varepsilon = \varepsilon_0$ the stiffnesses $\mathbf{D}$ and $\tilde{\mathbf{D}}$ coincide.

Under the refined scaling (13) the three-dimensional periodicity cell $\mathscr{Z} = Z \times (-x_3^+, x_3^+)$ of the original plate is for each ε homothetic to the three-dimensional cells $\varepsilon\mathscr{Y} = \varepsilon Y \times (-\varepsilon c, \varepsilon c)$, introduced in the three-dimensional asymptotic approach by Caillerie (1984, model $e \approx \varepsilon$) and Kohn and Vogelius (1984, model $a = 1$), cf. also Kalamkarov *et al.* (1987) and Section 3 in Lewiński (1991a). The scaling of the loading compensates the stiffness loss when ε tends to zero.

The problem $(\hat{P}_{HR}^{\varepsilon})$ defined by the refined scaling (13) and (14) amounts to finding $(\hat{w}^{\varepsilon}, \hat{\varphi}^{\varepsilon}) \in V(\Omega)$ such that

$$(\hat{P}_{HR}^{\varepsilon}) \left| \begin{array}{l} \displaystyle\int_{\Omega} \left[\hat{D}^{\alpha\beta\lambda\mu}\left(\frac{x}{\varepsilon}\right) \hat{\varphi}_{\lambda,\mu}^{\varepsilon} \psi_{\alpha,\beta} + \frac{1}{\varepsilon^2} \hat{H}^{\alpha\beta}\left(\frac{x}{\varepsilon}\right) (\hat{\varphi}_{\alpha}^{\varepsilon} + \hat{w}_{,\alpha}^{\varepsilon})(\psi_{\beta} + v_{,\beta}) \right] \mathrm{d}x = \int_{\Omega} \hat{q} v \, \mathrm{d}x \\ \text{for every } (v, \psi) \in V(\Omega). \end{array} \right. \tag{17}$$

It is worth noting that the $(\hat{P}_{HR}^{\varepsilon})$ problem turns out to be a direct Hencky–Reissner approximation of the bending part of the three-dimensional problem (P_{ε}) formulated in Lewiński (1991a, Section 3). Similar to the problem (P_{HR}), the problem $(\hat{P}_{HR}^{\varepsilon})$ is well-established for $\varepsilon > 0$.

4.2. *Asymptotic solution*

In this section a formal asymptotic analysis for solving the problem $(\hat{P}_{HR}^{\varepsilon})$ will be put forward. The rigorous justification of the method will be published in a separate paper.

The solution $(\hat{w}^{\varepsilon}, \hat{\varphi}^{\varepsilon})$ is looked for in the form

$$\hat{w}^\varepsilon = w^0(x)+\varepsilon w^1\left(x,\frac{x}{\varepsilon}\right)+\varepsilon^2 w^2\left(x,\frac{x}{\varepsilon}\right)+\cdots,$$

$$\hat{\varphi}^\varepsilon_\alpha = \varphi^0_\alpha(x)+\varepsilon\varphi^1_\alpha\left(x,\frac{x}{\varepsilon}\right)+\varepsilon^2\varphi^2_\alpha\left(x,\frac{x}{\varepsilon}\right)+\cdots, \tag{18}$$

where $(w^0,\varphi^0)\in V(\Omega)$ and $\varphi^k_\alpha(x,\cdot)$, $w^k(x,\cdot)$ are Y-periodic. Similarly we expand the trial fields

$$v = v^0(x)+\varepsilon v^1\left(x,\frac{x}{\varepsilon}\right)+\varepsilon^2 v^2\left(x,\frac{x}{\varepsilon}\right)+\cdots,$$

$$\psi_\alpha = \psi^0_\alpha(x)+\varepsilon\psi^1_\alpha\left(x,\frac{x}{\varepsilon}\right)+\varepsilon^2\psi^2_\alpha\left(x,\frac{x}{\varepsilon}\right)+\cdots, \tag{19}$$

where $(v^0,\psi^0)\in V(\Omega)$ and $v^k(x,\cdot)$, $\psi^k_\alpha(x,\cdot)$ are Y-periodic. Let us insert the expressions (18), (19) into (17) and require that the bilinear form given by its left-hand side does not grow to infinity when ε diminishes to zero. Hence we deduce that

$$\varphi^0_\alpha = -w^0_{,\alpha},\quad \psi^0_\alpha = -v^0_{,\alpha},$$
$$w^1 = w^1(x),\quad v^1 = v^1(x). \tag{20}$$

Therefore we assume that w^0, v^0 are of H^2_0 class. Let us define

$$\tilde{\varphi}_\alpha = \varphi^1_\alpha - w^1_{,\alpha},\quad \tilde{\psi}_\alpha = \psi^1_\alpha - v^1_{,\alpha}. \tag{21}$$

The main terms of eqn (17) assume the following form

$$\int_\Omega\left[\hat{D}^{\alpha\beta\lambda\mu}\left(\frac{x}{\varepsilon}\right)(k_{\lambda\mu}+\tilde{\varphi}_{\lambda|\mu})(-v^0_{,\alpha\beta}+\tilde{\psi}_{\alpha|\beta})\right.$$
$$\left.+\hat{H}^{\alpha\beta}\left(\frac{x}{\varepsilon}\right)(\tilde{\varphi}_\alpha+w^2_{|\alpha})(\tilde{\psi}_\beta+v^2_{|\beta})\right]\mathrm{d}x = \int_\Omega \hat{q}v^0\,\mathrm{d}x+0(\varepsilon), \tag{22}$$

where $k_{\lambda\mu} = -w^0_{,\lambda\mu}$ and

$$f_{|\alpha} = \frac{\partial f(x,y)}{\partial y_\alpha}\left(y=\frac{x}{\varepsilon}\right).$$

Let us put $\tilde{\psi} = \mathbf{0}$ in eqn (22) and let ε go to zero. Making use of the averaging lemma (cf. Sanchez-Palencia, 1980, Chapter 5, p. 77) one finds

$$-\int_\Omega \hat{M}^{\alpha\beta}_h v^0_{,\alpha\beta}\,\mathrm{d}x = \int_\Omega qv^0\,\mathrm{d}x, \tag{23}$$

where

$$\hat{M}^{\alpha\beta}_h = \{\hat{D}^{\alpha\beta\lambda\mu}(y)[k_{\lambda\mu}(x)+\tilde{\varphi}_{\lambda|\mu}(x,y)]\}, \tag{24}$$

$$q = \{\hat{q}\},\quad \{\cdot\} = \frac{1}{|Y|}\int_Y(\cdot)\,\mathrm{d}y. \tag{25}$$

If one passes to zero with ε in eqn (22) and combines the resulting equation with eqn (23) one obtains

$$\int_\Omega \{\hat{D}^{\alpha\beta\lambda\mu}(y)[k_{\lambda\mu}(x)+\tilde{\varphi}_{\lambda|\mu}(x,y)]\tilde{\psi}_{\alpha|\beta}(x,y) + \hat{H}^{\alpha\beta}(y)[\tilde{\varphi}_\alpha(x,y)+w^2_{|\alpha}(x,y)][\tilde{\psi}_\beta(x,y)+v^2_{|\beta}(x,y)]\}\,\mathrm{d}x = 0. \quad (26)$$

Let us put $\tilde{\psi}_\alpha = \bar{\psi}(x)\psi_\alpha(y)$, $v^2 = \bar{v}(x)v(y)$, where $\psi_\alpha, v \in H^1_{\mathrm{per}}(Y)$; $\bar{\psi}, \bar{v} \in \mathscr{D}(\Omega)$. Hence one arrives at the local equations of the form

$$\{\hat{D}^{\alpha\beta\lambda\mu}(y)[k_{\lambda\mu}(x)+\tilde{\varphi}_{\lambda|\mu}(x,y)]\psi_{\alpha|\beta}(y)+\hat{H}^{\alpha\beta}(y)[\tilde{\varphi}_\alpha(x,y)+w^2_{|\alpha}(x,y)]\psi_\beta(y)\} = 0, \quad \text{for every} \quad \psi \in [H^1_{\mathrm{per}}(Y)]^2; \quad (27)$$

$$\{\hat{H}^{\alpha\beta}(y)[\tilde{\varphi}_\alpha(x,y)+w^2_{|\alpha}(x,y)]v_{|\beta}(y)\} = 0 \quad \text{for every} \quad v \in H^1_{\mathrm{per}}(Y). \quad (28)$$

Since both eqns (27), (28) are linear one can put their solution in the form

$$w^2 = W^{(\gamma\delta)}(y)k_{\gamma\delta}(x), \quad \tilde{\varphi}_\lambda = \Psi^{(\gamma\delta)}_\lambda(y)k_{\gamma\delta}(x). \quad (29)$$

The auxiliary fields: $W^{(\gamma\delta)} = W^{(\delta\gamma)} \in H^1_{\mathrm{per}}(Y)$, $\Psi^{(\gamma\delta)}_\lambda = \Psi^{(\delta\gamma)}_\lambda \in H^1_{\mathrm{per}}(Y)$ are solutions to the following local problem:

$$(P^H_{\mathrm{loc}}) \left| \begin{array}{l} \text{find } (W^{(\gamma\delta)}, \Psi^{(\gamma\delta)}) \in W_{\mathrm{per}}(Y) = H^1_{\mathrm{per}}(Y)\times[H^1_{\mathrm{per}}(Y)]^2 \text{ such that} \\ \{\hat{D}^{\alpha\beta\lambda\mu}(y)\Psi^{(\gamma\delta)}_{\lambda|\mu}\psi_{\alpha|\beta}+\hat{H}^{\alpha\beta}(y)(\Psi^{(\gamma\delta)}_\alpha+W^{(\gamma\delta)}_{|\alpha})(\psi_\beta+v_{|\beta})\} = -\{\hat{D}^{\alpha\beta\gamma\delta}(y)\psi_{\alpha|\beta}\} \quad (30) \\ \text{for every } (v,\psi)\in W_{\mathrm{per}}(Y). \end{array} \right.$$

Upon substitution of the second part of formulae (29) into definition (24) of the averaged moments, one finds the homogenized constitutive relationship

$$\hat{M}^{\alpha\beta}_h = \hat{D}^{\alpha\beta\lambda\mu}_h k_{\lambda\mu}, \quad (31)$$

the homogenized effective bending stiffnesses being defined by

$$\hat{D}^{\alpha\beta\delta\gamma}_h = \{\hat{D}^{\alpha\beta\lambda\mu}(y)[\delta^\delta_\lambda\delta^\gamma_\mu+\Psi^{(\delta\gamma)}_{\lambda|\mu}(y)]\}. \quad (32)$$

Note first that the mathematical structure of the local problem (P^H_{loc}) is similar to that of the initial problem (P_{HR}). The differences between them lie only in the boundary conditions and in the linear forms at the right-hand sides. Thus according to Lax–Milgram lemma the solution $\Psi^{(\alpha\beta)}$, $W^{(\alpha\beta)}$ exists. The fields $\Psi^{(\alpha\beta)}$ are determined uniquely while the fields $W^{(\alpha\beta)}$ are determined up to an additive constant.

Note, secondly, that the local problem (P^H_{loc}) coincides with that found in Section 5 of Lewiński (1991c) via stipulating the Hencky–Reissner constraints upon the three-dimensional Caillerie–Kohn–Vogelius local problem. Moreover, formula (32) is identical to formula (36) in Lewiński (1991c). Therefore, the proof of symmetrical properties

$$\hat{D}^{\alpha\beta\gamma\delta}_h = \hat{D}^{\gamma\delta\alpha\beta}_h = \hat{D}^{\beta\alpha\gamma\delta}_h = \hat{D}^{\beta\alpha\delta\gamma}_h \quad (33)$$

and positive definiteness of the tensor $\hat{\mathbf{D}}_h$ reported in Lewiński (1991c, Section 5) holds good.

We conclude that the homogenized problem of the Kirchhoff form:

(P_{hom}) | find $w^0 \in \mathring{H}^2_0(\Omega)$ such that the variational eqn (23) with the effective constitutive relationship (31) is fulfilled for each $v^0 \in \mathring{H}^2_0(\Omega)$

is uniquely solvable.

Let us return now to the problem $(\hat{P}^{\varepsilon}_{HR})$. The averaged couple resultants related to the original plate are determined according to the second part of eqn (52) from Lewiński (1991b), namely

$$M_h^{\alpha\beta} = \varepsilon^3 \hat{M}_h^{\alpha\beta}. \tag{34}$$

Thus the overall constitutive relationships are

$$M_h^{\alpha\beta} = D_{:h}^{\alpha\beta\lambda\mu} k_{\lambda\mu}, \quad D_{:h}^{\alpha\beta\lambda\mu} = \varepsilon^3 \hat{D}_h^{\alpha\beta\lambda\mu}; \tag{35}$$

compare eqns (50, part 2), (51) from Lewiński (1991b) and formula (36) from Lewiński (1991c). The loads referred to the middle plane are given by eqn (54, part 1) from Lewiński (1991b), hence

$$q^* = \varepsilon^3 q(x). \tag{36}$$

Thus the virtual work equation that replaces the variational equation of the problem (P_{HR}) reads

$$-\int_\Omega M_h^{\alpha\beta} v^0_{,\alpha\beta}\, \mathrm{d}x = \int_\Omega q^* v^0\, \mathrm{d}x \quad \text{for every} \quad v^0 \in \mathring{H}^2_0(\Omega). \tag{37}$$

After finding the field w^0 one can evaluate the distribution of the couple resultants by the formula

$$M^{\alpha\beta}(x) = \varepsilon^3 \hat{D}^{\alpha\beta\lambda\mu}\left(\frac{x}{\varepsilon}\right)\left[\delta^\gamma_\lambda \delta^\delta_\mu + \Psi^{(\gamma\delta)}_{\lambda|\mu}\left(y = \frac{x}{\varepsilon}\right)\right] k_{\gamma\delta}(x). \tag{38}$$

Note that averaging the above formula results in the overall constitutive relation (35, part 1).

4.3. *On relation to the three-dimensional asymptotic approach of Caillerie–Kohn–Vogelius*

Considered in this paper the asymptotic method takes as a departure point the two-dimensional problem (P_{HR}), being the Hencky–Reissner approximation of the original three-dimensional problem of the elastic layer clamped along its lateral surface. The asymptotic analysis that starts from this three-dimensional formulation can be performed by appropriate formation of the ε-indexed family of three-dimensional problems. From the point of view of the criterion of similarity of length scales (1) the constructions of Caillerie (1984, for $e \approx \varepsilon$) and Kohn and Vogelius (1984, 1985, for $a = 1$) can be regarded as physically justified. The asymptotic analysis based upon this ε-family of problems leads up to the three-dimensional local problems and to the Kirchhoff-type model for the effective plate.

Imposition of the Kirchhoff constraints upon the solutions to these local problems results in the formulae of Duvaut (1976) provided that the quantities $c^{\alpha\beta} = C^{33\alpha\beta}/C^{3333}$ do not depend on y_α, cf. Lewiński (1991d). Since the formulae of Duvaut have been found as a result of the two operations—the Kirchhoff approximation followed by homogenization—then, if $c^{\alpha\beta} = c^{\alpha\beta}(y_3)$ one can say that the above-mentioned operations commute, cf. Diagram 1, Lewiński (1991c).

If in the place of Kirchhoff constraints the Hencky–Reissner-like constraints are stipulated, and if homogenization of (P_{HR}) problem is based upon the in-plane scaling (10) then it turns out that the property of commutativeness is lost. First, the effective equations

Diagram 1.

3D(P)
the original problem

↓

3D(P_ε)

asymptotic analysis based on scaling (1) preserving similarity between length scales ↓	Reissner–Hencky reduction of the transverse dimension of the plate ↓
{3D(P^2_{loc}) posed on $\mathscr{Y}$, 2D(P_{hom}) posed on Ω}	2D($\hat{P}^\varepsilon_{HR}$)
Reissner–Hencky reduction of the transverse dimension of the cell $\mathscr{Y}$; $c^{\alpha\beta} = c^{\alpha\beta}(y_1)$ ↓	asymptotic analysis based on the scaling (13) preserving similarity between length scales
{2D(P^H_{loc}) posed on Y, 2D(P_{hom}) posed on Ω} ←	←

assume the form of Hencky–Reissner type and even after imposing constraints on rotations: $\varphi^0 = -\nabla w^0$, one arrives at the Kirchhoff plate with bending stiffnesses different than those following from imposing Hencky–Reissner constraints upon the Caillerie–Kohn–Vogelius local problem.

Using the refined scaling (14) creates a relationship between the three-dimensional asymptotic analyses. The local problem (P^H_{loc}) found in Section 4.2 coincides with that found in Section 5 of Lewiński (1991c) as a result of imposing Hencky–Reissner constraints on the Caillerie–Kohn–Vogelius local problem. Thus, if appropriately performed, the operations of homogenization and Hencky–Reissner reduction of the transverse plate dimension are commutative, see Diagram 1.

Diagram 1 is closed, which suggests that its right-hand path is justified, provided the left-hand one can be gone through. Therefore, the results of Sections 4.1 and 4.2 should rather be used in the case when the three-dimensional cell of periodicity has a shape of a moderately thick plate. It is insufficient to require only that the whole plate is "moderately thick".

5. ON TORSIONAL STIFFNESS OF PLATES WITH ONE FAMILY OF STIFFENERS

The formulae for the effective stiffnesses $D^{\alpha\alpha\beta\beta}$ of isotropic plates with thickness varying periodically in one direction are incontrovertible, cf. remarks in the Introduction. The aim of this section is to clear up the problem of evaluation of the effective torsional stiffness D^{1212} of such plates.

5.1. *General expressions*

Consider the plate made of an isotropic elastic material, the thickness of which varies periodically in x_1 direction and alternately assumes the values $2h_1$ or $2h_2$, cf. Fig. 2a. The periodicity cell $\mathscr{Z}$ can be viewed as two-dimensional and the plane basic cell Z reduces to the interval $(0, Z_1)$. The rescaled cell $\mathscr{Y} = \mathscr{Z}/\varepsilon_0$; $Y_1 = a$, Z_2 is arbitrary, cf. Fig. 2a and b.

Let us recall that according to Duvaut (1976) homogenizing Kirchhoff equations results in the following formula for the effective torsional stiffness, cf. eqn (38, part 2) of Lewiński (1991c) for $\varepsilon = \varepsilon_0$:

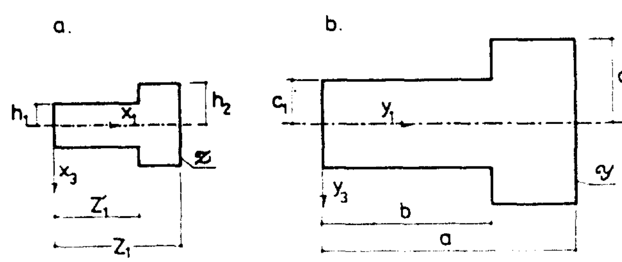

Fig. 2. The basic cell $\mathscr{Z}$ and the rescaled cell $\mathscr{Y}$; $c_i = h_i/\varepsilon_0$, $b = Z'_1/\varepsilon_0$, $a = Z_1/\varepsilon_0$.

$$D_h^{1212} = \varepsilon_0^3\{\hat{D}^{1212}\}, \quad \{\cdot\} = \frac{1}{a}\int_0^a (\cdot)\,\mathrm{d}y_1. \tag{39}$$

On the other hand, homogenizing P_{HR} with the use of the in-plane scaling (10) leads to

$$D_H^{1212} = \varepsilon_0^3\{(\hat{D}^{1212})^{-1}\}^{-1}, \tag{40}$$

cf. eqn (39, part 2) of Lewiński (1991c), given that $\varepsilon = \varepsilon_0$.

By homogenizing the same problem with the help of the refined scaling (13), or by following the left-hand side of Diagram 1, one obtains for $\varepsilon = \varepsilon_0$

$$D_{:h}^{1212} = \varepsilon_0^3\{\hat{D}^{1212}(1+\Psi_{2|1}^{(12)})\} \tag{41}$$

see eqn (52) of Lewiński (1991c).

For various dimensions $(h_x, Z_1', Z_1) = \varepsilon_0(c_x, b, a)$, and for the case of the isotropic material several results obtained with the help of the above formulae have been reported in Section 7 of Lewiński (1991c). The predictions of formula (41) have been found numerically. In this section we shall derive a closed form formula which interrelates the stiffness $D_{:h}^{1212}$ with the geometrical data: a, b, c_x.

5.2. *Derivation of the formula* $D_{:h}^{1212}(a, b, c_x)$

We shall find the exact solution to the problem $(P_{\mathrm{loc}}^{\prime\prime})$ for $(\gamma\delta) = (12)$ in the case of the considered isotropic plate with the piece-wise constant thickness.

The bending and transverse shearing stiffnesses are

$$\hat{D}^{1212}(y_1) = \begin{cases} \frac{2}{3}G_s(c_1)^3 & \text{for} \quad y_1 \in I = (0, b) \\ \frac{2}{3}G_s(c_2)^3 & \text{for} \quad y_1 \in (b, a) \end{cases} \tag{42}$$

$$\hat{H}^{22}(y_1) = \begin{cases} 2\kappa c_1 G_s & \text{for} \quad y_1 \in I \\ 2\kappa c_2 G_s & \text{otherwise.} \end{cases} \tag{43}$$

We shall assume $\kappa = 5/6$; $2G_s = E/(1+\nu)$, E represents Young's modulus, ν is the Poisson ratio. Let us define a function

$$\alpha(y_1) = (\hat{H}^{22}/\hat{D}^{1212})^{1/2} = \begin{cases} (5/2)^{1/2}/c_1 & \text{for} \quad y_1 \in I \\ (5/2)^{1/2}/c_2 & \text{for} \quad y_1 \in (b, a). \end{cases} \tag{44}$$

The function $\Psi_2^{(12)}$ satisfies the following ordinary differential equation [cf. eqn (46, part 3) of Lewiński (1991c)]:

$$\frac{\mathrm{d}^2\Psi_2^{(12)}}{\mathrm{d}y_1^2} - \alpha^2(y_1)\Psi_2^{(12)} = 0, \tag{45}$$

the switching conditions at $y_1 = b$

$$\begin{aligned} \Psi_2^{(12)}(b-0) &= \Psi_2^{(12)}(b+0), \\ M^{12(12)}(b-0) &= M^{12(12)}(b+0), \end{aligned} \tag{46}$$

and the periodicity conditions

$$\Psi_2^{(12)}(0) = \Psi_2^{(12)}(a),$$
$$M^{12(12)}(0) = M^{12(12)}(a), \tag{47}$$

where [cf. eqn (47, part 2) of Lewiński (1991c)]

$$M^{12(12)} = \hat{D}^{1212}\left(\frac{d\Psi_2^{(12)}}{dy_1}+1\right). \tag{48}$$

Let us introduce the following notation:

$$D_2 = \tfrac{2}{3}G_s(h_2)^3,\quad \sigma = c_1/c_2,\quad \lambda = (5/2)^{1/2}b/c_1,\quad \omega = a/b,$$
$$\xi = y_1/b,\quad u(\xi) = \frac{1}{b}\Psi_2^{(12)}(y_1). \tag{49}$$

Thus $D_2 = \varepsilon_0^3\hat{D}^{1212}(y_1)$ for $y_1 \in (b, a)$.

The solution u can be represented in the form

$$u(\xi) = \begin{cases} C_1\exp(-\lambda\xi)+B_1\exp(-\lambda(1-\xi)) & \text{for } \xi\in[0,1] \\ C_2\exp(-\lambda\sigma(\xi-1))+B_2\exp(-\lambda\sigma(\omega-\xi)) & \text{for } \xi\in[1,\omega]. \end{cases} \tag{50}$$

Using the switching and periodicity conditions one finds $B_\alpha = -C_\alpha$ and

$$C_1 = \frac{\sigma^3-1}{\sigma\lambda}\,\frac{1-\exp(-\lambda\sigma(\omega-1))}{f(\sigma,\omega,\lambda)},$$
$$C_2 = -\frac{\sigma^3-1}{\sigma\lambda}\,\frac{1-\exp(-\lambda)}{f(\sigma,\omega,\lambda)}, \tag{51}$$

where

$$f(\sigma,\omega,\lambda) = \sigma^2(1+\exp(-\lambda))(1-\exp(-\lambda\sigma(\omega-1))) + (1-\exp(-\lambda))(1+\exp(-\lambda\sigma(\omega-1))). \tag{52}$$

According to formula (41) one finds

$$D_{:h}^{1212}/D_2 = \frac{1}{\omega}(\sigma^3+\omega-1)-\frac{(1-\sigma^3)^2}{\omega\sigma\lambda}\,g(\sigma,\omega,\lambda) \tag{53}$$

where

$$g(\sigma,\omega,\lambda) = \frac{2}{\sigma^2\coth\frac{\lambda}{2}+\coth\frac{\lambda(\omega-1)\sigma}{2}}. \tag{54}$$

The torsional stiffness due to Duvaut, eqn (39), reads

$$D_h^{1212}/D_2 = \frac{1}{\omega}(\sigma^3+\omega-1), \tag{55}$$

while formula (40) assumes the form

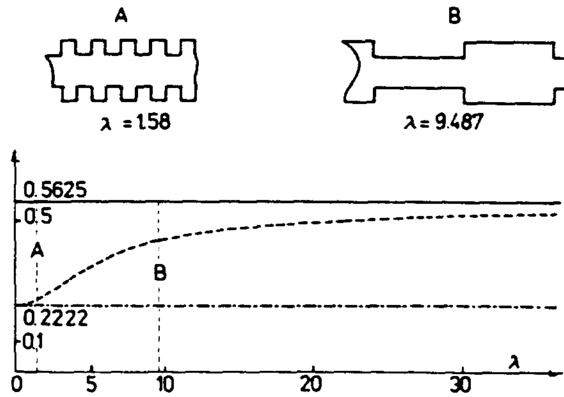

Fig. 3. The effective torsional stiffness of plates from the λ-indexed family of fixed $\sigma = 0.5$, $\omega = 2$. Exemplary plates A, B have the same volume and differ in λ. The solid line represents the upper asymptotics (55) of Duvaut; the lower asymptotics: —·—·—·—·— is determined by eqn (56); the curve ——————— denotes $D_{:h}^{1212}/D_2$ given by (53) and (54).

$$D_H^{1212}/D_2 = \frac{\omega\sigma^3}{\sigma^3(\omega-1)+1}. \tag{56}$$

One can prove that formulae (53), (55) and (56) do not change the form if one substitutes: $c_1 \to c_2$, $c_2 \to c_1$, $b \to a-b$. Thus these formulae are not sensitive to the choice of the cell of periodicity $\mathscr{Y}$. The results set up previously in Tables 7.3a–7.3d, 7.4, 7.5 from Lewiński (1991c) can be obtained by the formulae given above.

5.3. *A family of plates for which* c_x *and* ω *are fixed*

Let us fix c_1, c_2 and ω. Then $\lambda = (5/2)^{1/2}(a/\omega c_1)$ is proportional to a. Note that D_h^{1212}, D_H^{1212} do not depend on λ. One can prove that

$$D_{:h}^{1212} \xrightarrow[\lambda\to 0]{} D_H^{1212}, \quad D_{:h}^{1212} \xrightarrow[\lambda\to\infty]{} D_h^{1212} \tag{57}$$

and

$$D_H^{1212} \leqslant D_{:h}^{1212} \leqslant D_h^{1212}. \tag{58}$$

If c_x and ω are fixed the mean thickness of the plates of various λ is constant, and thus the volumes of all these plates are the same (cf. Figs. 3–5)—the exemplary plates marked A and B are of the same volume. The plates B can be built from plates A by appropriate rearranging of stiffeners, but without adding new material. For such a family of plates the result of Duvaut (eqn (55)) provides the upper asymptotics for $D_{:h}^{1212}(\lambda)$, while formula

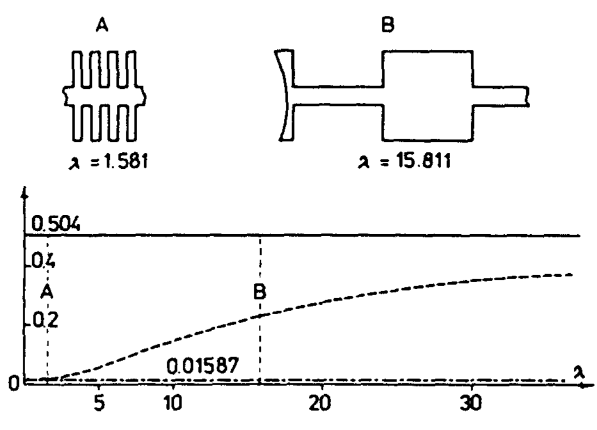

Fig. 4. The effective torsional stiffnesses versus λ of plates of fixed $\sigma = 0.2$, $\omega = 2$. Denotations as in Fig. 3.

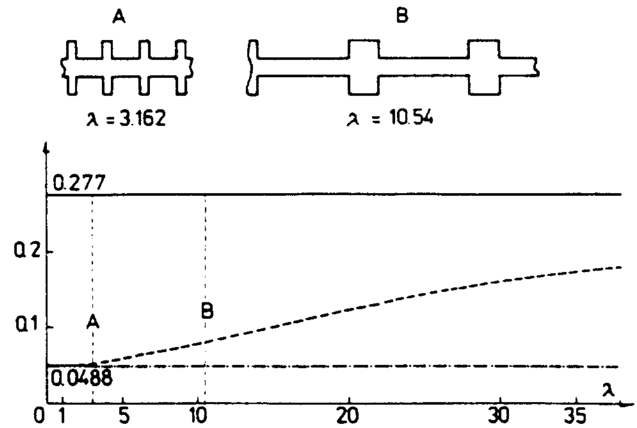

Fig. 5. The effective torsional stiffnesses of plates of fixed $\sigma = 1/3$, $\omega = 4/3$. Denotations as in Fig. 3.

(56) determines its lower asymptotics concerning three cases: $\sigma = 0.5$, $\omega = 2$ (Fig. 3), $\sigma = 0.2$, $\omega = 2$ (Fig. 4), and $\sigma = 1/3$, $\omega = 4/3$ (Fig. 5). One can show that the speed of convergence of $D_{;h}^{1212}(\lambda)$ to D_h^{1212} strongly depends on the value of the ratio $\sigma = c_1/c_2$. For $\sigma \gg 1$ or $\sigma \ll 1$ this convergence can be very slow.

5.4. *Family of plates of fixed length of the period, fixed volume and indexed by the ratio* $\sigma = c_1/c_2$

Let us fix a and ω. We assume that the mean thickness

$$c = \frac{1}{a}[bc_1 + (a-b)c_2] \tag{59}$$

is kept fixed. Thus the parameters σ and λ are interrelated by

$$\sigma\lambda = \left(\frac{5}{2}\right)^{1/2} \frac{a}{c} \frac{\sigma + \omega - 1}{\omega^2}. \tag{60}$$

Bearing in mind the above relation and the definition of D_2:

$$D_2/E(Z_1)^3 = \frac{5}{6}\left(\frac{5}{2}\right)^{1/2} \frac{1}{1+\nu} \frac{1}{(\omega\sigma\lambda)^3} \tag{61}$$

one can find the nondimensional stiffnesses

$$\bar{D}_{;h}(\sigma, \omega, a/c) = D_{;h}^{1212}/E(Z_1)^3,$$

$$\bar{D}_h(\sigma, \omega, a/c) = D_h^{1212}/E(Z_1)^3, \quad \bar{D}_H(\sigma, \omega, a/c) = D_H^{1212}/E(Z_1)^3. \tag{62}$$

In the case $\omega = 2$ ($b = a/2$) the above functions are invariant under the change $\sigma \to 1/\sigma$; thus then it is sufficient to restrict the domain of σ to the interval $(0, 1]$. In this interval the

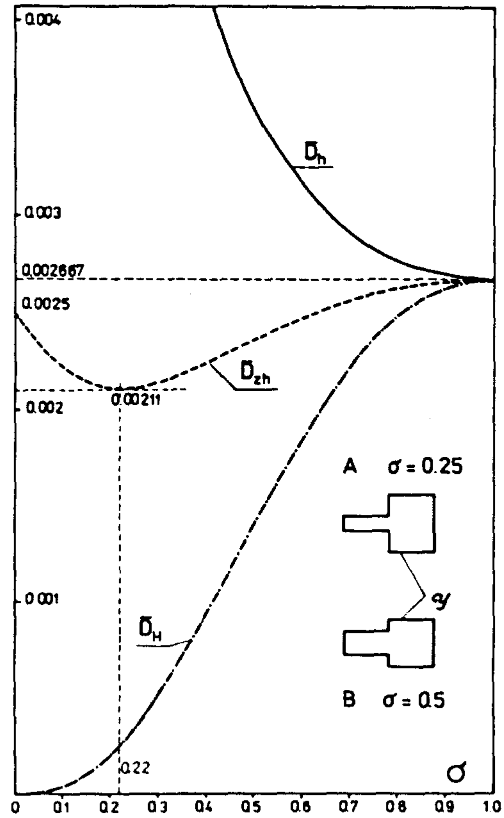

Fig. 6. The effective torsional stiffnesses versus σ for plates composed of cells $\mathscr{Y}$ such that: $\nu = 0.25$, $\omega = 2$, $a/c = 5$, $c_1+c_2 = 0.4a$. For displayed shapes of $\mathscr{Y}\sigma = 0.25$ (A) and $\sigma = 0.5$ (B). All cells are of the same volume. It could not be shown that for $\sigma = 0$, $\bar{D}_h = 0.010667$.

function $\bar{D}_{:h}(\cdot, 2, a/c)$ assumes a minimum in a certain internal point, cf. Fig. 6 ($a/c = 5$, $c_1+c_2 = 0.4a$) and Fig. 7 ($a/c = 2$, $c_1+c_2 = a$). The maximum value of this function is achieved for $\sigma = 1$ or for the case when the stiffeners are absent. The available results of the Kohn and Vogelius three-dimensional approach (1984, 1985) have been placed in Fig. 7. These results show that in the case considered the predictions of Duvaut (1976) are completely inaccurate.

5.5. *Torsional stiffness of the gridwork*

Let us fix a, b, c_2. Then D_2 and ω are also fixed. Let $c_1 \to 0$ ($\sigma \to 0$). Then $\lambda = (5/2)^{1/2}\cdot a/(\omega c_1) \to \infty$, but $\lambda\sigma = (5/2)^{1/2}\cdot a/(\omega c_2)$ is fixed. The limiting values of (53), (55), (56) are

$$D_h^{1212}/D_2 \to (\omega-1)/\omega, \quad D_H^{1212}/D_2 \to 0,$$

$$D_{:h}^{1212}/D_2 \to (\omega-1)/\omega - \frac{1}{\beta}\,\mathrm{th}\left(\beta\frac{\omega-1}{\omega}\right), \tag{63}$$

where $\beta = 0.5(5/2)^{1/2}\cdot a/c_2$. The results above estimate the torsional stiffness of the gridwork composed of independent beams, cf. Fig. 8a. Note that the limiting value of D_h^{1212}/D_2 does not depend on the ratio $(a-b)/c_2$.

Let us keep a and c_2 fixed and let $b \to 0$ ($\omega \to \infty$). Then

$$D_{:h}^{1212}/D_2 \to 1 - \frac{\mathrm{th}\,\beta}{\beta}, \quad D_h^{1212}/D_2 \to 1. \tag{64}$$

The above results refer to the case of the gridwork of Fig. 8b. The formula of Duvaut disregards that the plate in Fig. 8b is composed of independent beams.

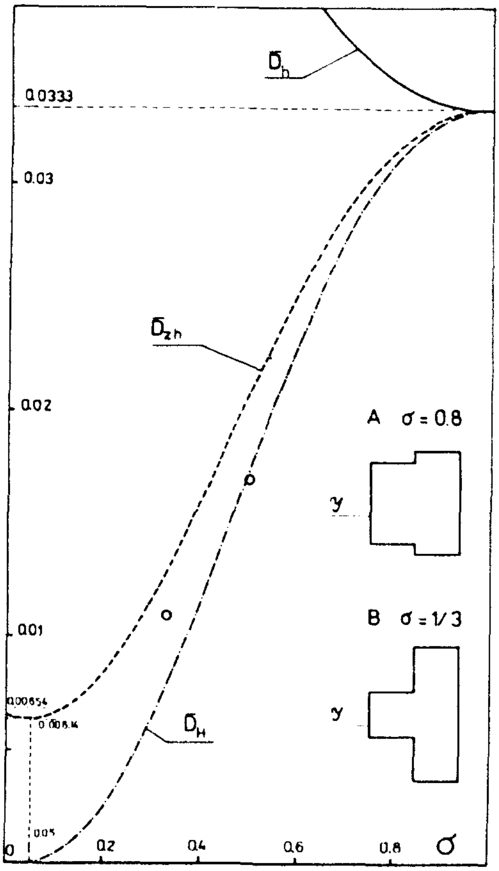

Fig. 7. The effective torsional stiffnesses versus σ for plates composed of cells such that: $\nu = 0.25$, $\omega = 2$, $a/c = 2$, $c_1 + c_2 = a$. The exemplary cells A and B of different σ are of the same volume. The circles denote the results obtained via a three-dimensional approach of Kohn and Vogelius (1984). It could not be shown that for $\sigma = 0$, $\bar{D}_h = 0.1333$.

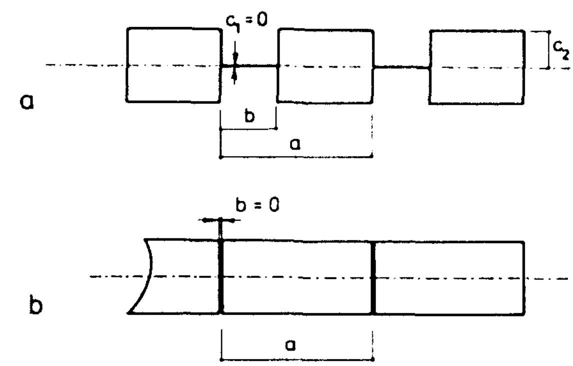

Fig. 8. The gridwork composed of independent beams. In case (b) the distance between beams is zero.

6. FINAL REMARKS

(1) Developed recently the regularized relaxed formulations of the optimization problems of inhomogeneous plates (cf. Olhoff *et al.*, 1981; Rozvany *et al.*, 1982; Lur'e and Cherkaev, 1986; Bendsøe, 1987) involve Duvaut's formulae for effective stiffnesses of plates with unidirectionally periodic structure. These formulae have widely been used in the literature although their very narrow range of applicability has been recognized (cf. Kohn and Vogelius, 1984; Remark 1 of Section 6.3 in Lur'e and Cherkaev, 1986). The present paper shows how to improve these formulae and, consequently, how to improve the relaxed

formulations. The formulae for $D_{zh}^{\alpha\alpha\beta\beta}$ do not have to be changed whilst the formula (55) should be replaced by the formula (53) which can be rewritten as follows:

$$D_{zh}^{1212} = \frac{2G_s}{3Z_1}\left[b_1h_1^3+b_2h_2^3-\frac{(h_1^3-h_2^3)^2}{\mu h_1^2\coth\left(\mu\frac{b_1}{h_1}\right)+\mu h_2^2\coth\left(\mu\frac{b_2}{h_2}\right)}\right]. \tag{65}$$

Here $\mu = 1/2(3\kappa)^{1/2}$ (for $\kappa = 5/6$, $\mu = (5/8)^{1/2}$), $b_1 = Z'_1$, $b_2 = Z_1-Z'_1$, cf. Fig. 2.

(2) The present paper shows that the refined scaling (5) is superior to the in-plane scaling (2) in the case of plates with periodically varying stiffnesses. By virtue of the refined scaling the formulae for effective stiffnesses become sensitive to the transverse shape of the periodicity cell. In previous studies on smearing-out fissures in plates, in-plane scaling has been used (cf. Lewiński and Telega, 1988b; Telega and Lewiński, 1988). As a consequence the formulae for effective stiffnesses obtained in this manner have turned out to be insensitive to the distance between fissures measured with respect to the plate thickness. The refined scaling improves these results and the effective stiffnesses become functions of the crack density, which is characteristic for cracked plates, cf. Hashin (1985).

(3) One can conjecture that the refined scaling (5) should result in reasonable homogenization formulae not only in the theory of Reissner–Hencky plates with periodic structure but also in other periodic problems in which their own natural length scales are present. A typical representative is a micropolar medium. If one assumes the in-plane scaling in the problem of homogenization of properties of periodic micropolar medium, one arrives at the homogenized model of micropolar-type (cf. Bytner and Gambin, 1986). On the other hand, if one imposes the refined scaling then in the homogenized problem all length scales vanish and the homogenized problem becomes a classical one. The length scales remain in the local problem which assumes the form of the original problem of the micropolar medium.

(4) The asymptotic analysis based upon the refined scaling (eqn (13)) has led us to a homogenized model of the Kirchhoff type and to a local problem similar to the initial problem of Reissner–Hencky form. However, the initial problem could assume the form of one of the known improved plate models. Then the refined scaling would lead, also in this case, to the Kirchhoff form of the homogenized model and to the local problem of mathematical structure of the initial improved theory of plates. Thus the more difficult and accurate the initial model is, the more hard and exact the local analysis is to be analyzed. Also, in the most general case when the departure point is three-dimensional (which can be viewed as a perfect improved model that imposes no constraints) the homogenized model turns out to be of the Kirchhoff type and the local problem is posed on a three-dimensional cell of periodicity (cf. Caillerie, 1984 model $e \approx \varepsilon$; Kohn and Vogelius, 1984, 1985, case $a = 1$). Thus in every case the using of the refined scaling transmits the structure of the initial problem to the local analysis. The question whether this observation concerns shells with periodic structure is still open.

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